Note's On Goldbach's Conjecture

by Todd R. Kueny, Sr.
todd@lexigraph "dot" com
September 26th, 2010 - *draft 01*

Goldbach's Conjecture, which asserts that all positive even integers >= 4 can be expressed as the sum of two primes, presents some interesting puzzles. This paper lays out some insights I have developed.

We begin with a process of generating even numbers via an *odd diagonal* table. Assume a table of the form shown in Ex. 1.

The *odd diagonal table* is a table with the the series of consecutive odd numbers beginning with 3, e.g., 3,5,7,9,..., extending to infinity, running down the right diagonal. We well refer to these odd numbers the *odd diagonal roots* of the table.

ASSERTION: The intersection of every row and column under the odd diagonal is an even number.

Each number, being the sum of the odd number at the top of the column (*column root*) and the odd number at the right of the row (*row root*), must be even as shown in Ex. 1 by the intersection of the two red boxes, i.e., $o_1 = 2k + 1$, $o_2 = 2m + 1$ hence $o_1 + o_2 = 2m + 2k + 2$ which is always even.

3														
8	**5**													
10	12	**7**												
12	14	16	**9**											
14	16	18	20	**11**										
16	18	20	22	24	**13**									
18	20	22	24	26	28	**15**								
20	22	24	26	28	30	32	**17**							
22	24	26	28	30	32	34	36	**19**						
24	26	28	30	32	34	36	38	40	**21**					
26	28	30	32	34	36	38	40	42	44	**23**				
28	30	32	34	36	38	40	42	44	46	48	**25**			
30	32	34	36	38	40	42	44	46	48	50	52	**27**		
32	34	36	38	40	42	44	46	48	50	52	54	56	**29**	
34	36	38	40	42	44	46	48	50	52	54	56	58	60	**31**

Ex. 1

We assert that at the intersection of the first column and second row, beginning under the three (**3**), all even numbers starting with eight (8) are generated as we move down subsequent rows (advancing along the odd diagonal), i.e., within each column we step by $odd_{n+1} - odd_n = 2$, subsequently each even is advanced by the corresponding amount.

As we shift right one column the same assertion applies with the exception that the first number under the diagonal (five in the second column, and so on) advances by two and the sum at the intersection advances by two.

The purpose of this table is to show that the intersection of each row and column where each row and column root represents an odd integer, it is sufficient, in the context of the *odd diagonal table,* to generate all even numbers.

The blue *counter-diagonal* in Ex. 1 shows that the *odd diagonal table* not only generates all the even number but it redundantly generates each even number starting with 12.

For each odd number *n* we see that the length of the *counter-diagonal* is equal to the *position* of *n* on the odd diagonal of the table and is also a reflection about the row, thus 11 is the 5th odd number with 4 prior odd numbers. The length of the *counter-diagonal* (where each value is 22) excluding 11 itself is equal to 4. Note that 11 doubled is also 22 making the total length 5.

We assert that this *counter-diagonal*, called **d**, represents the sum of *all* possible unique *pairs* of odd integers that may add up to twice the odd integer root (including all primes as a subset of odd integers) on the odd diagonal, i.e., in the case of 11 we see that 22 (twice 11 or 11 + 11) must therefore be made up of 11+11, 13 + 9, 15 + 7, 17 + 5, and 19 + 3.

We now define the set **D** as the set of each diagonal representing an even value greater than 12. Each diagonal is defined as starting in the left-most column under **3** and extending up and left by one in both directions consecutively until there is either no odd root entry or we reach an odd diagonal root. The *root* of the diagonal is the diagonal value divided by 2. The odd diagonal root at the upper-left of the diagonal, when doubled, is equal to the values in the diagonal, e.g., 22 = 2 x 11 and 11 + 11, and is therefore the pair is included in the diagonal. Each value in the diagonal represents unique pairs of odd integer sums. For example, with 11, there are five values of 22 in the diagonal represented left to right by 11 + 11, 13 + 9, 15 + 7, 17 + 5, and 19 + 3.

Counter-diagonals relative to even roots, e.g., a diagonal based on 16, have no corresponding odd diagonal root because 16 divided by two is even.

We can now redefine the binary Goldbach Conjecture as follows in terms of counter diagonals:

> For each even *n* associated with each *counter-diagonal* **d** in **D** there exists
> at least one pair of odd prime integers (*a* and *b*) whose sum is *n.* *(1)*

For example we can see by inspection in the case of *n*=22 that 11+11, 17 + 5, and 19 + 3 confirm the conjecture but 15 + 7 and, 13 + 9 do not because 9 and 15 are not prime. Clearly this table also displays *all* possible sum combinations of two numbers we are interested in.

For our interest in (1) we must show that for our odd diagonal table, when we eliminate all non-prime numbers and their corresponding respective pairs of sums from their respective counter diagonals, there will be at least one prime odd number pair remaining in each member of **D**.

In Ex. 2 we show that by *striking* both the row and column associated with each non-prime we can eliminate from consideration all non-prime numbers for pair sums. We do this by always eliminating both the row *and* column associated the non-prime odd root extending to infinity as necessary vertically. We show this in Ex. 2 as a line through both the row and column. All rows and columns so struck are subsequently removed from all consideration because the represent multiples of lower odd primes.

3														
8	**5**													
10	12	**7**												
12	14	16	**9**											
14	16	18	20	**11**										
16	18	20	22	24	**13**									
18	20	22	24	26	28	**15**								
20	22	24	26	28	30	32	**17**							
22	24	26	28	30	32	34	36	**19**						
24	26	28	30	32	34	36	38	40	**21**					
26	28	30	32	34	36	38	40	42	44	**23**				
28	30	32	34	36	38	40	42	44	46	48	**25**			
30	32	34	36	38	40	42	44	46	48	50	52	**27**		
32	34	36	38	40	42	44	46	48	50	52	54	56	**29**	
34	36	38	40	42	44	46	48	50	52	54	56	58	60	**31**

Ex. 2

While this effectively compacts the table we will not show that here for clarity. We can select which numbers are prime along the odd diagonal root by any sieve process (note that we consider all even numbers to be already eliminated from the sieve), e.g., eliminating all odd multiples of 3, 5, 7 and so on.

For example in Ex 2. by striking the row and column for 9, 15, 21, ... we can no longer consider multiples of 3, 5, 7, ... in any sum remaining in any *counter-diagonal* in the table.

By striking all non-primes we are left with only the sums of two primes and their even results for all even numbers greater than 8. We do not consider the even prime two (2) for this proof.

Let us reexamine Ex. 2. Note the green counter-diagonal that represents all possible two-number combinations that add to 30. Since we are interested in all *counter-diagonals* let us warp the table in Ex. 2 so that each row corresponds to a counter-diagonal.

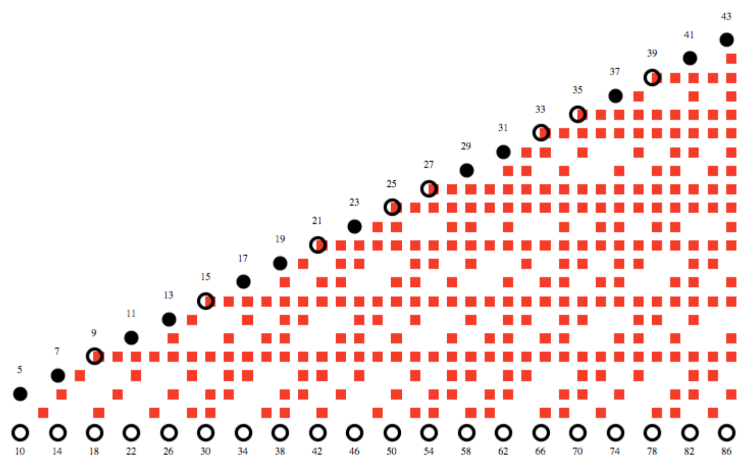
10, **5**
12, 12
14, 14, **7**
16, 16, 16
18, 18, 18, **9**
20, 20, 20, 20
22, 22, 22, 22, **11**
24, 24, 24, 24, 24
26, 26, 26, 26, 26, **13**
28, 28, 28, 28, 28, 28
30, 30, 30, 30, 30, 30, **15**
32, 32, 32, 32, 32, 32, 32
34, 34, 34, 34, 34, 34, 34, **17**
36, 36, 36, 36, 36, 36, 36, 36
38, 38, 38, 38, 38, 38, 38, 38, **19**

And, including the strikes:

10, **5**

- , 12
14, - , **7**
16, 16, -
- , 18, 18, **-**
20, - , 20, -
22, 22, - , - , **11**
- , 24, 24, - , 24
26, - , 26, - , - , **13**
- , 28, - , - , 28, -
- , - , 30, - , 30, 30, **-**
32, - , - , - , - , 32, -
34, 34, - , - , 34, - , - , **17**
- , 36, 36, - , - , 36, - , 36
- , - , 38, - , - , - , - , - , **19**

Representing this graphically as Ex. 3 below and using one square per numeric value (or strike) from our table above we see



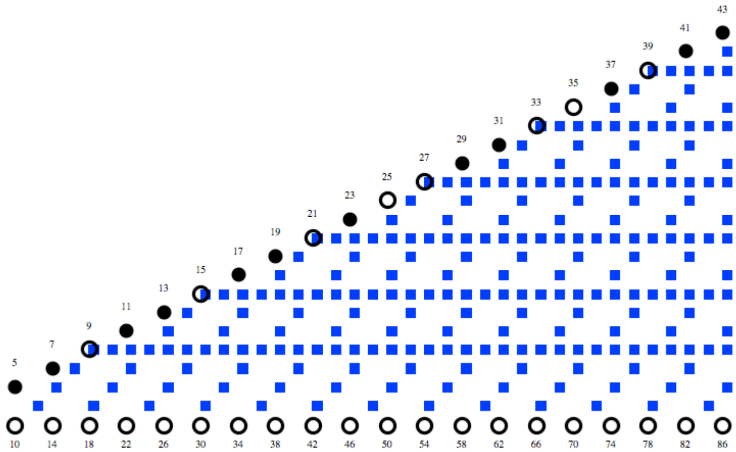
where white (actually transparent to the white background of the sheet) represents an entry on the *counter-diagonal* that is the sum of two primes and red represents an entry that is not.

(Some discussion of motivation here. At this point all that I have done is replace the locations of prime addends with nothing and the non-prime addends with a colored square. I did this initially in Mathematica to try and create a better representation of the primes and the associated addends. Once I was able to observe this table it became there was some kind of structure - but it was to complicated to understand in and of itself. My next idea was to realize that the primes are actually a series of individual sieves: 2, 3, 5 and so on. So I starting looking first at the geometric structure of only the 3 sieve, then the 5 sieve, then the two combined, and so on.)

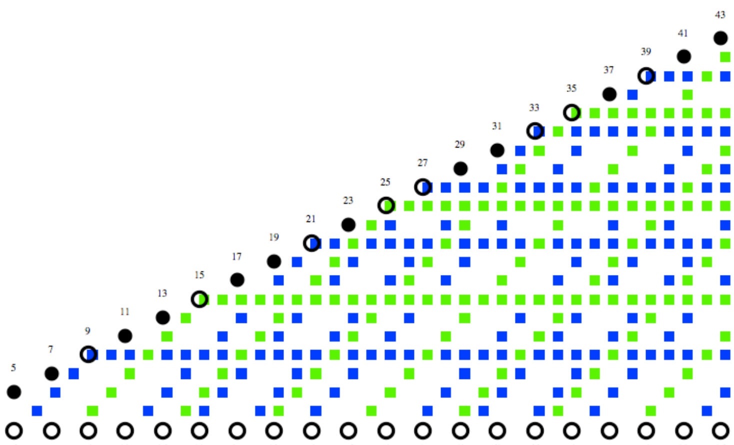
We now convert our notion of non prime strikes to a notion of sieving (iterating over) all values made up of non prime addends from our columns (the original *counter-diagonals*) based on the diagonal of odd integers. We use a color mark to indicate each cell is a factor of some odd integer.

We first sieve for all factors of 3, i.e., a cell becomes blue where $x > 3$ and Modulo(x, 3) is zero.
We move down the diagonal to the next odd integer that has not been sieved, in this case 5.
We first sieve for all factors of 5, i.e., a cell becomes blue where $x > 5$ and Modulo(x, 5) is zero.
We move down the diagonal to the next odd integer that has not been sieved, in this case 7.
We first sieve for all factors of 7, i.e., a cell becomes blue where $x > 7$ and Modulo(x, 7) is zero.
We move down the diagonal to the next odd integer that has not been sieved, in this case 11.
We first sieve for all factors of 11, i.e., a cell becomes blue where $x > 11$ and Modulo(x, 11) is zero.
and so on to infinity.

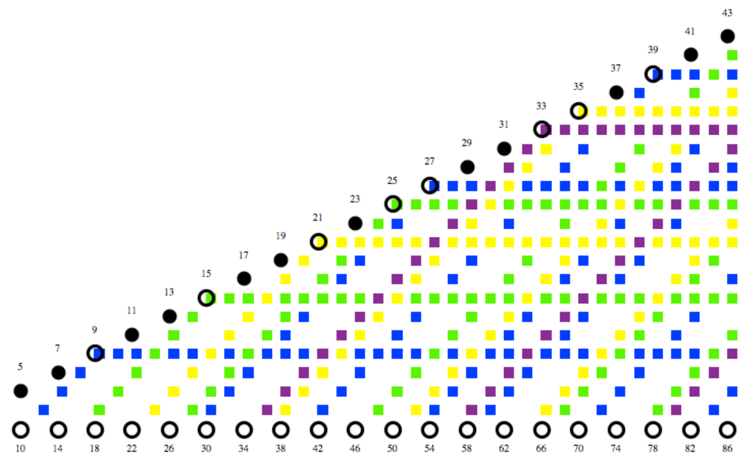
So the first iteration for factors of three as blue yields:



Superimposing the second iteration for factors of five as green *on top* yields:

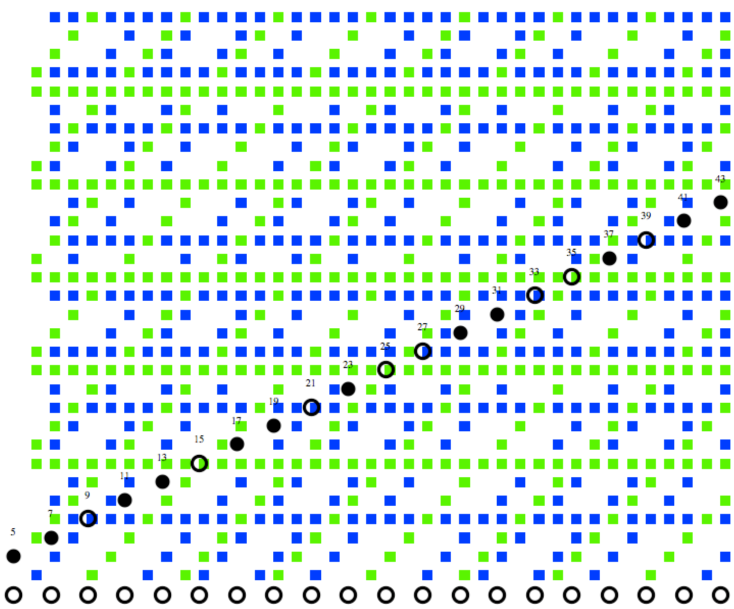


And so forth (here adding factors of seven as yellow and eleven as purple). This ultimately leaves the prime numbers and all prime addends clear of marks.



Now let us consider the placement of these structures on an X/Y grid with the origin at the lower left and let us not limit the extension of pattern of strikes to the diagonal showing the primes.

(The motivation here is that while you can consider the primes alone there is actually a larger structure involved. This became apparent with this graphic.)



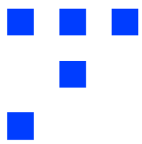
We note that the primes appear in this structure at an angle of rise=1, run=2 (1/Sqrt[5]) - we call this the *prime diagonal.* We see that primes appear within the center of each green and blue trapezoid and we observe that all trapezoids must begin on a non-prime. The relative harmonic behavior of 3 and 5 is seen starting with 15 - the first value where both 3 and 5 are common factors. The 3 sieve skips to 21 leaving 17 and 19. The five sieve skips to 25 leaving 23. We also see the period of the 3 and 5 trapezoids is 15.

We observe that under each diagonal each prime added is represented. If our sieves only include three and five then our addends are sieved only by three and five. Adding more sieves, say for seven, produce interesting results which well discuss subsequently to our discussion of tiling below.

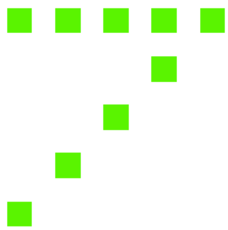
By adding subsequent iterations for new odd numbers and allowing our structure to grow to the left and upwards we create a pattern representing the sieving of all odd numbers.

Note that this structure naturally eliminates common factors, e.g., 9, as seen by the horizontal line extending leftward from 9.

Now let us examine the structure of the patterns presented in this larger structure. We can see from this structure that a 3 x 3 *tessellation* emerges

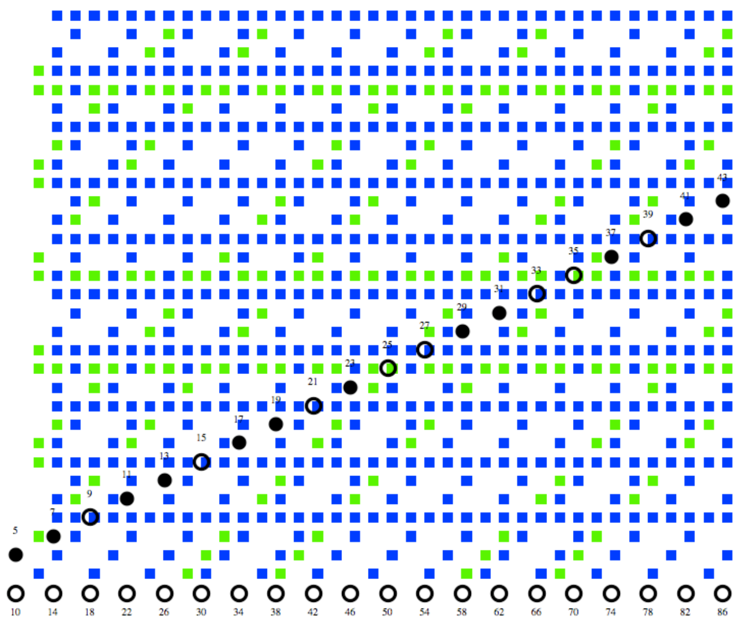


as well as a 5x5 tessellation.



In fact, all odd numbers in our sieve may be represented by extending this basic pattern horizontally and diagonally by the value of the odd number. Note that there is an extra row across the bottom where a horizontal strike would appear but does not (it would correspond in each case to the given odd row, e.g., there would be a row at three, five, seven, and so forth). The the perspective of the tessellation this can be ignored because it represent *fewer* strikes than the basic tessellations.

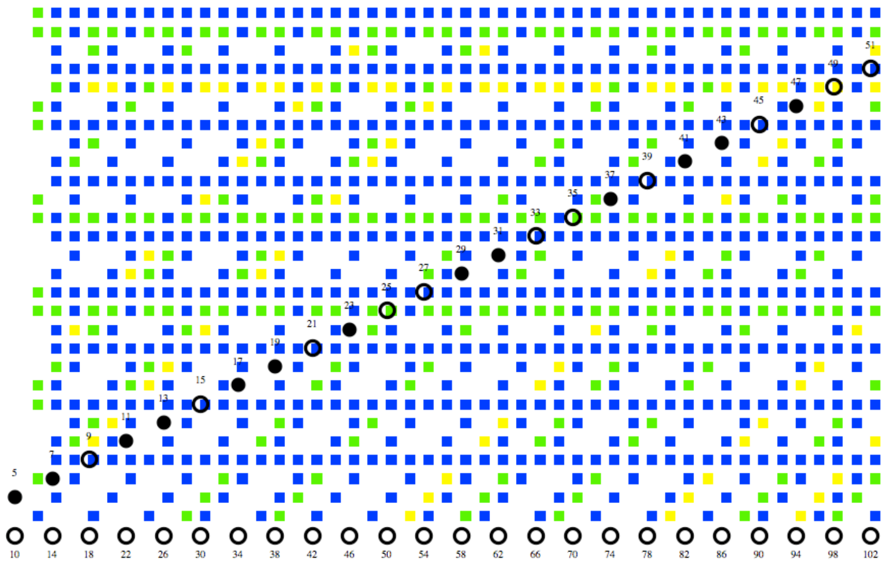
Now let us superimpose the blue tessellation for three *on top* of the green tessellation for five:



We observe that the green five tessellation serves either to "connect" two vertical elements of the blue three tessellation or to create a new component (one or two green squares and one blue square).

We can note that the *longest possible connection* between green and blue is a sequence of eight elements (two blue, one green, two blue, one green, and two blue or 2-1-2-1-2) as seen vertically between 21 and 23 on the prime diagonal. This connection is *bounded* by our combination of tessellating patterns - both in terms of the *period* and the relative *starting position*. We define the period of the tessellation as the number of cells required before the pattern repeats, i.e., for three and five the period is fifteen. (Note the relationship between the tessellating trapezoids and the underlying tiles.) In a general sense the relative positions of the two patterns is unimportant but their placement here must account the appearance of their respective multiples along the *prime diagonal*, i.e., the five tessellation must be positioned so that 15, 25, and 35 are sieved out.

Now we introduce the tessellation for seven (note that it is presented with the three and five tessellations *on top*).



We observe that this additional tessellation connects previously disconnected elements of the combined three and give tessellations.