Note's On Goldbach's Conjecture

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Goldbach's Conjecture, which asserts that all positive even integers >= 4 can be expressed as the sum of two primes, presents some interesting puzzles. This paper lays out some insights I have developed.

We begin with a process of generating even numbers via an *odd diagonal* table. Assume a table of the form shown in Ex. 1.

The *odd diagonal table* is a table with the the series of consecutive odd numbers beginning with 3, e.g., 3,5,7,9,..., extending to infinity, running down the right diagonal. We well refer to these odd numbers the *odd diagonal roots* of the table.

ASSERTION: The intersection of every row and column under the odd diagonal is an even number.

Each number, being the sum of the odd number at the top of the column (*column root*) and the odd number at the right of the row (*row root*), must be even as shown in Ex. 1 by the intersection of the two red boxes, i.e., $o_1 = 2k + 1$, $o_2 = 2m + 1$ hence $o_1 + o_2 = 2m + 2k + 2$ which is always even.

3														
8	**5**													
10	12	**7**												
12	14	16	**9**											
14	16	18	20	**11**										
16	18	20	22	24	**13**									
18	20	22	24	26	28	**15**								
20	22	24	26	28	30	32	**17**							
22	24	26	28	30	32	34	36	**19**						
24	26	28	30	32	34	36	38	40	**21**					
26	28	30	32	34	36	38	40	42	44	**23**				
28	30	32	34	36	38	40	42	44	46	48	**25**			
30	32	34	36	38	40	42	44	46	48	50	52	**27**		
32	34	36	38	40	42	44	46	48	50	52	54	56	**29**	
34	36	38	40	42	44	46	48	50	52	54	56	58	60	**31**

Ex. 1

We assert that at the intersection of the first column and second row, beginning under the three (**3**), all even numbers starting with eight (8) are generated as we move down subsequent rows (advancing along the odd diagonal), i.e., within each column we step by $odd_{n+1} - odd_n = 2$, subsequently each even is advanced by the corresponding amount.

As we shift right one column the same assertion applies with the exception that the first number under the diagonal (five in the second column, and so on) advances by two and the sum at the intersection advances by two.

The purpose of this table is to show that the intersection of each row and column where each row and column root represents an odd integer, it is sufficient, in the context of the *odd diagonal table,* to generate all even numbers.

The blue *counter-diagonal* in Ex. 1 shows that the *odd diagonal table* not only generates all the even number but it redundantly generates each even number starting with 12.

For each odd number *n* we see that the length of the *counter-diagonal* is equal to the *position* of *n* on the odd diagonal of the table and is also a reflection about the row, thus 11 is the 5th odd number with 4 prior odd numbers. The length of the *counter-diagonal* (where each value is 22) excluding 11 itself is equal to 4. Note that 11 doubled is also 22 making the total length 5.

We assert that this *counter-diagonal*, called **d**, represents the sum of *all* possible unique *pairs* of odd integers that may add up to twice the odd integer root (including all primes as a subset of odd integers) on the odd diagonal, i.e., in the case of 11 we see that 22 (twice 11 or 11 + 11) must therefore be made up of 11+11, 13 + 9, 15 + 7, 17 + 5, and 19 + 3.

We now define the set **D** as the set of each diagonal representing an even value greater than 12. Each diagonal is defined as starting in the left-most column under **3** and extending up and left by one in both directions consecutively until there is either no odd root entry or we reach an odd diagonal root. The *root* of the diagonal is the diagonal value divided by 2. The odd diagonal root at the upper-left of the diagonal, when doubled, is equal to the values in the diagonal, e.g., 22 = 2 x 11 and 11 + 11, and is therefore the pair is included in the diagonal. Each value in the diagonal represents unique pairs of odd integer sums. For example, with 11, there are five values of 22 in the diagonal represented left to right by 11 + 11, 13 + 9, 15 + 7, 17 + 5, and 19 + 3.

Counter-diagonals relative to even roots, e.g., a diagonal based on 16, have no corresponding odd diagonal root because 16 divided by two is even.

We can now redefine the binary Goldbach Conjecture as follows in terms of counter diagonals:

> For each even *n* associated with each *counter-diagonal* **d** in **D** there exists
> at least one pair of odd prime integers (*a* and *b*) whose sum is *n.* *(1)*

For example we can see by inspection in the case of *n*=22 that 11+11, 17 + 5, and 19 + 3 confirm the conjecture but 15 + 7 and, 13 + 9 do not because 9 and 15 are not prime. Clearly this table also displays *all* possible sum combinations of two numbers we are interested in.

For our interest in (1) we must show that for our odd diagonal table, when we eliminate all non-prime numbers and their corresponding respective pairs of sums from their respective counter diagonals, there will be at least one prime odd number pair remaining in each member of **D**.

In Ex. 2 we show that by *striking* both the row and column associated with each non-prime we can eliminate from consideration all non-prime numbers for pair sums. We do this by always eliminating both the row *and* column associated the non-prime odd root extending to infinity as necessary vertically. We show this in Ex. 2 as a line through both the row and column. All rows and columns so struck are subsequently removed from all consideration because the represent multiples of lower odd primes.

3														
8	**5**													
10	12	**7**												
~~12~~	~~14~~	~~16~~	~~**9**~~											
14	16	18	20	**11**										
16	18	20	22	24	**13**									
~~18~~	~~20~~	~~22~~	~~24~~	~~26~~	~~28~~	~~**15**~~								
20	22	24	26	28	30	32	**17**							
22	24	26	28	30	32	34	36	**19**						
~~24~~	~~26~~	~~28~~	~~30~~	~~32~~	~~34~~	~~36~~	~~38~~	~~40~~	~~**21**~~					
26	28	30	32	34	36	38	40	42	44	**23**				
~~28~~	~~30~~	~~32~~	~~34~~	~~36~~	~~38~~	~~40~~	~~42~~	~~44~~	~~46~~	~~48~~	~~**25**~~			
~~30~~	~~32~~	~~34~~	~~36~~	~~38~~	~~40~~	~~42~~	~~44~~	~~46~~	~~48~~	~~50~~	~~52~~	~~**27**~~		
32	34	36	38	40	42	44	46	48	50	52	54	56	**29**	
34	36	38	40	42	44	46	48	50	52	54	56	58	60	**31**

Ex. 2

While this effectively compacts the table we will not show that here for clarity. We can select which numbers are prime along the odd diagonal root by any sieve process (note that we consider all even numbers to be already eliminated from the sieve), e.g., eliminating all odd multiples of 3, 5, 7 and so on.

For example in Ex 2. by striking the row and column for 9, 15, 21, ... we can no longer consider multiples of 3, 5, 7, ... in any sum remaining in any *counter-diagonal* in the table.

By striking all non-primes we are left with only the sums of two primes and their even results for all even numbers greater than 8. We do not consider the even prime two (2) for this proof.

Let us reexamine Ex. 2. Note the green counter-diagonal that represents all possible two-number combinations that add to 30. Since we are interested in all *counter-diagonals* let us warp the table in Ex. 2 so that each row corresponds to a counter-diagonal.

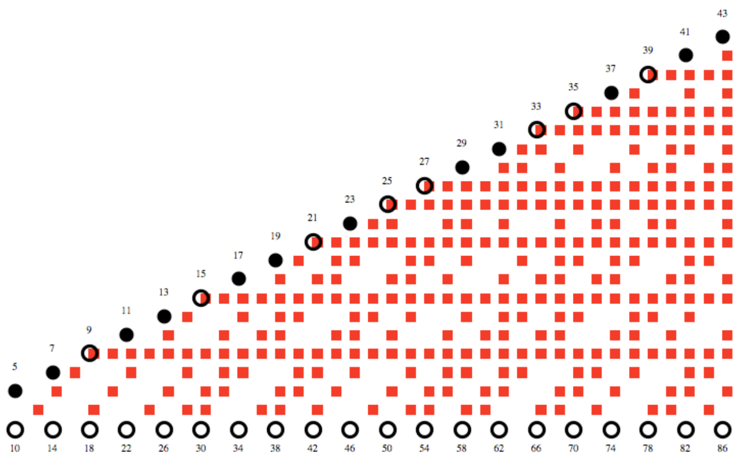
10, **5**
12, 12
14, 14, **7**
16, 16, 16
18, 18, 18, **9**
20, 20, 20, 20
22, 22, 22, 22, **11**
24, 24, 24, 24, 24
26, 26, 26, 26, 26, **13** Ex. 3
28, 28, 28, 28, 28, 28
30, 30, 30, 30, 30, 30, **15**
32, 32, 32, 32, 32, 32, 32
34, 34, 34, 34, 34, 34, 34, **17**
36, 36, 36, 36, 36, 36, 36, 36
38, 38, 38, 38, 38, 38, 38, 38, **19**

And, including the strikes:

10, **5**

- , 12
14, - , **7**
16, 16, -
- , 18, 18, **-**
20, - , 20, -
22, 22, - , - , **11**
- , 24, 24, - , 24 Ex. 4
26, - , 26, - , - , **13**
- , 28, - , - , 28, -
- , - , 30, - , 30, 30, **-**
32, - , - , - , - , 32, -
34, 34, - , - , 34, - , - , **17**
- , 36, 36, - , - , 36, - , 36
- , - , 38, - , - , - , - , - , **19**

Representing this graphically as Ex. 5 below and using one square per numeric value (or strike) from our table above we see



Ex. 5

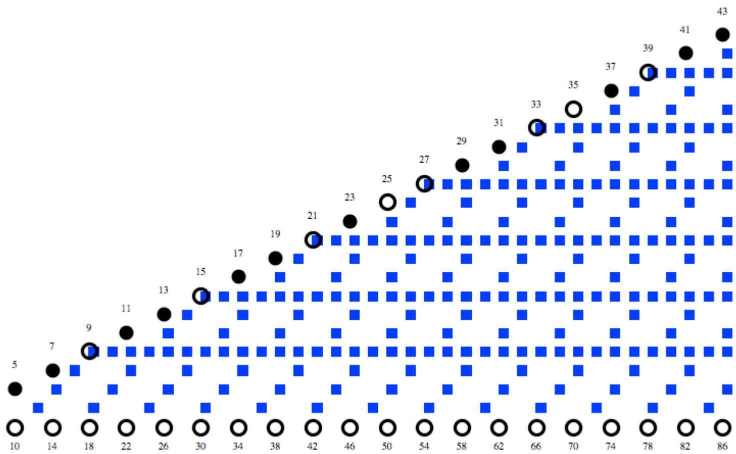
where white (actually transparent to the white background of the sheet) represents an entry on the *counter-diagonal* that is the sum of two primes and red represents an entry that is not.

(Some discussion of motivation here. At this point all that I have done is replace the locations of prime addends with nothing and the non-prime addends with a colored square. I did this initially in Mathematica to try and create a better representation of the primes and the associated addends. Once I was able to observe this table it became there was some kind of structure - but it was to complicated to understand in and of itself. My next idea was to realize that the primes are actually a series of individual sieves: 2, 3, 5 and so on. So I starting looking first at the geometric structure of only the 3 sieve, then the 5 sieve, then the two combined, and so on.)

We now convert our notion of non prime strikes to a notion of sieving (iterating over) all values made up of non prime addends from our columns (the original *counter-diagonals*) based on the diagonal of odd integers. We use a color mark to indicate each cell is a factor of some odd integer.

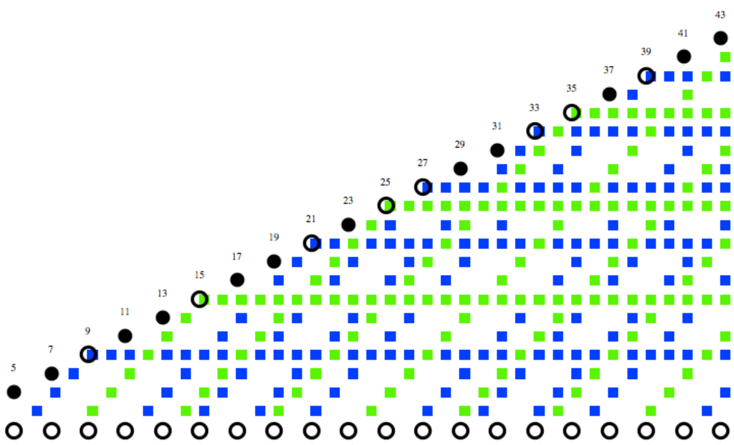
We first sieve for all factors of 3, i.e., a cell becomes blue where $x > 3$ and Modulo(x, 3) is zero.
We move down the diagonal to the next odd integer that has not been sieved, in this case 5.
We first sieve for all factors of 5, i.e., a cell becomes blue where $x > 5$ and Modulo(x, 5) is zero.
We move down the diagonal to the next odd integer that has not been sieved, in this case 7.
We first sieve for all factors of 7, i.e., a cell becomes blue where $x > 7$ and Modulo(x, 7) is zero.
We move down the diagonal to the next odd integer that has not been sieved, in this case 11.
We first sieve for all factors of 11, i.e., a cell becomes blue where $x > 11$ and Modulo(x, 11) is zero.
and so on to infinity.

So the first iteration for factors of three as blue yields:



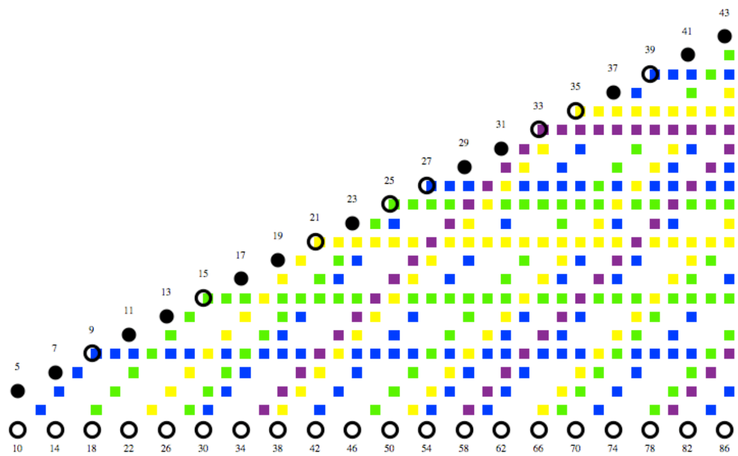
Ex. 6

Superimposing the second iteration for factors of five as green *on top* yields:



Ex. 7

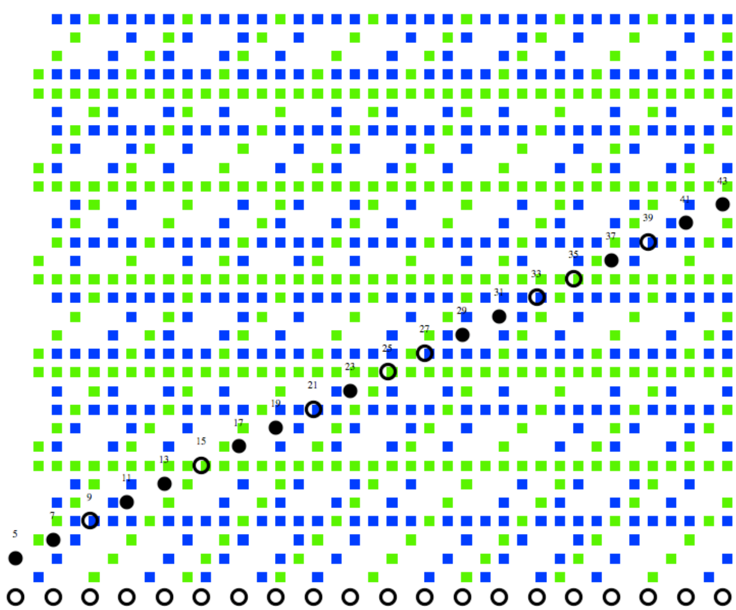
And so forth (here adding factors of seven as yellow and eleven as purple). This ultimately leaves the prime numbers and all prime addends clear of marks.



Ex. 8

Now let us consider the placement of these structures on an X/Y grid with the origin at the lower left and let us not limit the extension of pattern of strikes to the diagonal showing the primes.

(The motivation here is that while you can consider the primes alone there is actually a larger structure involved. This became apparent with this graphic.)



Ex. 9

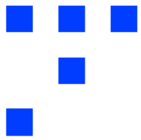
We note that the primes appear in this structure at an angle of rise=1, run=2 (1/Sqrt[5]) - we call this the *prime diagonal*. We see that primes appear within the center of each green and blue trapezoid and we observe that all trapezoids must begin on a non-prime. The relative harmonic behavior of 3 and 5 is seen starting with 15 - the first value where both 3 and 5 are common factors. The 3 sieve skips to 21 leaving 17 and 19. The five sieve skips to 25 leaving 23. We also see the period of the 3 and 5 trapezoids is 15.

We observe that under each diagonal each prime added is represented. If our sieves only include three and five then our addends are sieved only by three and five. Adding more sieves, say for seven, produce interesting results which well discuss subsequently to our discussion of tiling below.

By adding subsequent iterations for new odd numbers and allowing our structure to grow to the left and upwards we create a pattern representing the sieving of all odd numbers.

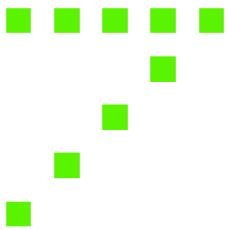
Note that this structure naturally eliminates common factors, e.g., 9, as seen by the horizontal line extending leftward from 9.

Now let us examine the structure of the patterns presented in this larger structure. We can see from this structure that a 3 x 3 *tessellation* emerges



Ex. 10

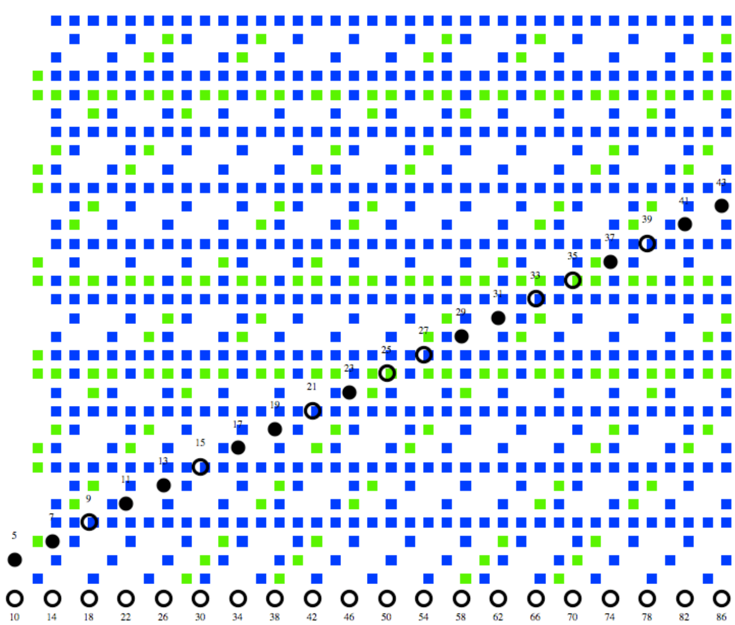
as well as a 5x5 tessellation.



Ex. 11

In fact, all odd numbers in our sieve may be represented by extending this basic pattern horizontally and diagonally by the value of the odd number. Note that there is an extra row across the bottom where a horizontal strike would appear but does not (it would correspond in each case to the given odd row, e.g., there would be a row at three, five, seven, and so forth). The the perspective of the tessellation this can be ignored because it represent *fewer* strikes than the basic tessellations.

Now let us superimpose the blue tessellation for three *on top* of the green tessellation for five:

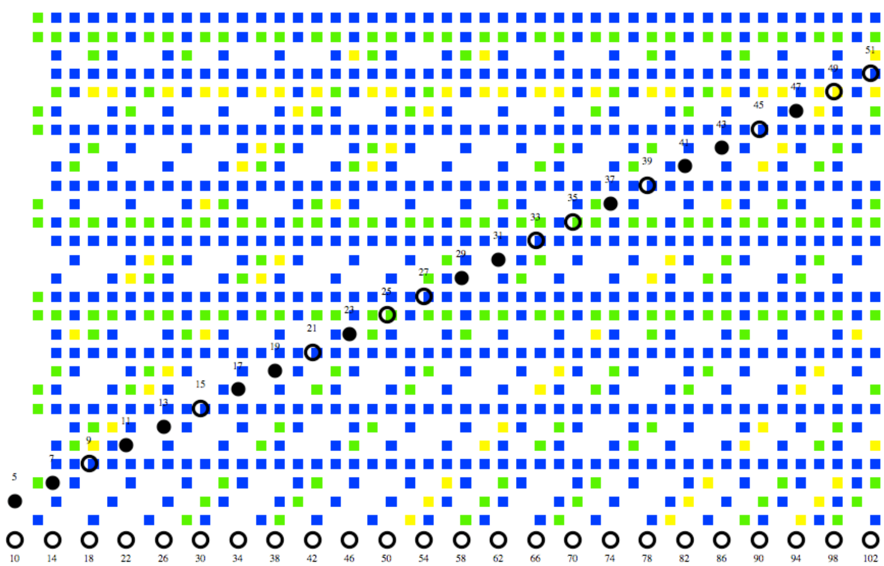


Ex. 12

In Ex. 12 we observe that the green five tessellation serves either to "connect" two vertical elements of the blue three tessellation or to create a new component (one or two green squares and one blue square).

We can note that the *longest possible connection* between green and blue is a sequence of eight elements (two blue, one green, two blue, one green, and two blue or 2-1-2-1-2) as seen vertically between 21 and 23 on the prime diagonal. This connection is *bounded* by our combination of tessellating patterns - both in terms of the *period* and the relative *starting position*. We define the period of the tessellation as the number of cells required before the pattern repeats, i.e., for three and five the period is fifteen. (Note the relationship between the tessellating trapezoids and the underlying tiles.) In a general sense the relative positions of the two patterns is unimportant but their placement here must account the appearance of their respective multiples along the *prime diagonal*, i.e., the five tessellation must be positioned so that 15, 25, and 35 are sieved out.

Now we introduce the tessellation for seven (note that it is presented with the three and five tessellations *on top*).



Ex. 13

We observe that this additional tessellation connects previously disconnected elements of the combined three and give tessellations.

One interesting point is that the introduction of the seven tessellation does not impact the underlying structure until the diagonal from 52 along the bottom (between the open circles for 50 and 54) and the previously open circle on the prime diagonal for 49. The reason for this is that seven (squared to 49) does not participate *uniquely* as an addend until 54 (49 + 3 - the smallest prime). So effectively this yellow diagonal is a demarcation indicating when seven begins to effect the structure of the tessellation. This same principle is true for the squares of all primes added to the tessellation. Similarly by comparing Ex. 7 and Ex. 9 we see the same is true for 5 - it does not participate until 25.

These examples serve to make an additional point. The 3 x 3 tessellation is by far the most *dominant*. It alone eliminates 5/9 of all the possible cells which could be prime. When representing these tessellations we generally place the 3 x 3 on top in order to demonstrate the function of subsequent tessellatoins in terms of connecting 3 x 3 elements. When we place other tessellations on top of the 3 x 3 we reveal the underlying structure of those tessellations.

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Boxed Primes

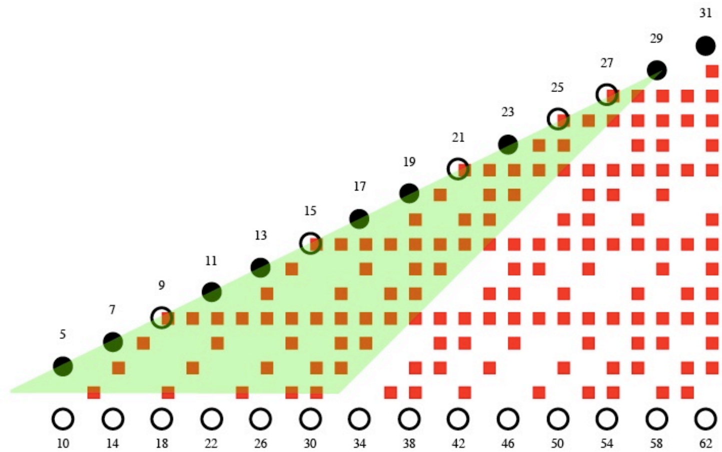
Let us now consider the issue of "Boxed Primes". Our definition of *boxed primes* is as follows: Take a prime P and double it, e.g., P[n] where n=7 so the 7th prime is 17 which, when doubled, is 34.

(I call this a "box" because its a square box of primes, say from 3..17, running across the top and down the side. At each row/column intersection you sum the corresponding row/column prime.)

Take all unique arrangements of two primes from P[2] (3) to P[7] (17): {3,3}, {3,5}, {5,5}, ..., {13, 17}, {17,17} and add the pairs together, e.g., {3,3} = 3 + 3 = 6 and eliminate duplicates, i.e., {5,5} and {3,7} both add up to 10, to create a set of unique sums.

For 17, as an example, the permutations of the pairs yields all evens >= 6 and less than or equal 34 except 32. For 19 the permutations of pairs yields all evens >= 6 and less than or equal to 38. Similarly for 109 and 218.

The example below demonstrates graphically the boxed prime 29 on our tessellating tile surface:

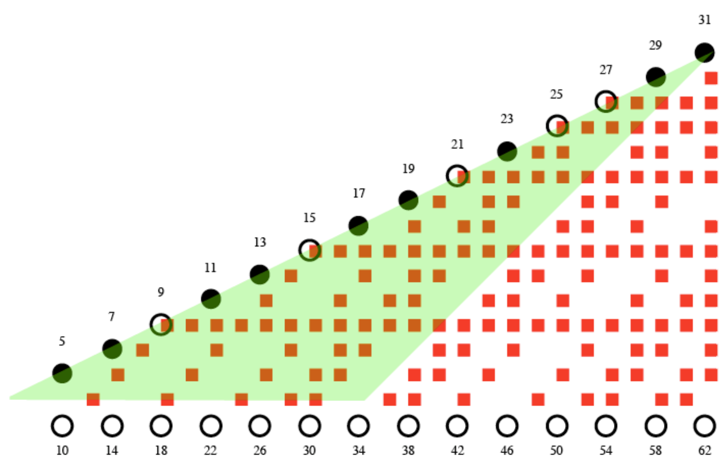


Ex. 14

Here we use a semi-transparent green triangle to represent the boxed primes. The base (bottom) of the triangle running from above 10 to above 32 represents even values for which we have exhausted the tessellated representation of prime addends, i.e., all possible addends for these values participate in the prime box. We say this because the entire vertical column of addends appears *inside* the triangle.

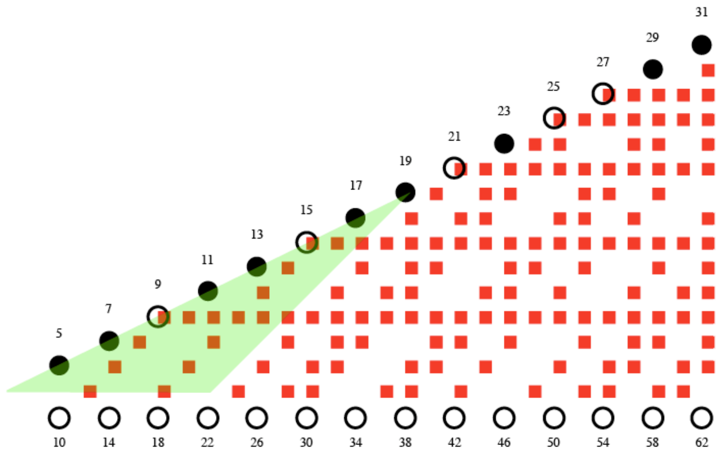
The right diagonal from 32 (between the open circle for 30 and 34) to the prime 29 (black dot) at the top represents prime addends where we consider only a portion of the possible addends for a given even number - in particular the addends closest to 1/2 the even value. As before we are considering only the value *inside* the triangle. For example, the value 22 along the top above 44 at the bottom. Since the entire column from the lower diagonal of the triangle to the upper diagonal of the triangle is populated by red squares 44 does not appear in the boxed primes for 29. Similarly for 25/50 and 28/56 the region from the lower to upper diagonal is blocked by red squares. Note that for all primes, like 19, the column always has an open cell because the topmost value, e.g., 19, is always included in the triangle.

If we extend the triangle proportionally to 31 we see that 44 is now included in the box because the lower diagonal now passes through an open cell in the column for 22/44. However 28 remains blocked.



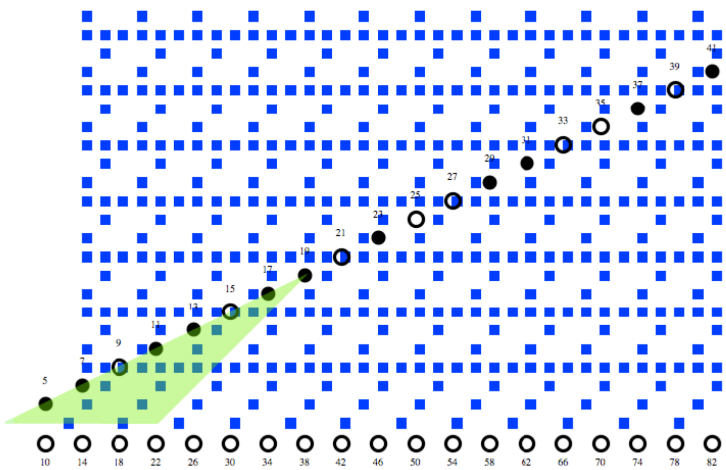
Ex. 15

Extend the triangle proportionally in the other direction to 19 we see that for all even values there is at least one cell which is not blocked in each column included in the area inside the triangle.



Ex. 16

Let us now make some more detailed observations about the general structure of Ex. 13. relative to Boxed Primes.

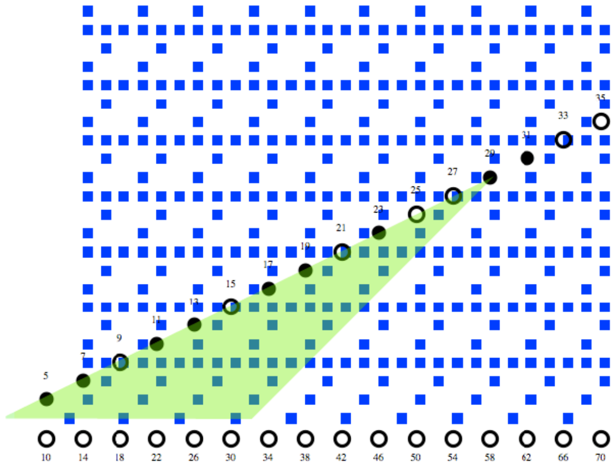


Ex. 17

We notice that for the three sieve that each position which could contain a prime (non-multiple of three locations) there are two blue squares *both above* and *below* in that column. In addition, we see that this pattern repeats for each tessellation within that column - both above and below the *prime diagonal*.

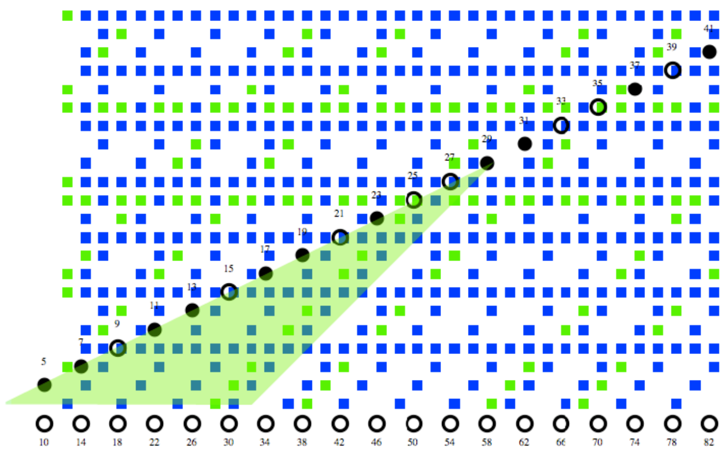
We also observe that the position of the green triangle for all "boxed primes" below 25 will have the same result whether or not any higher-valued prime sieve is also applied.

So what is the role of a subsequent sieve application, .e.g., five sieve? Without the five sieve our example produces incorrect results - in this case, for example, there is a missing green diagonal (see Ex. 13) extending from 25 and 25 itself is "prime" in the sense that no colored cell covers it. Similarly the cell below 27 is open.



Ex. 18

If we now apply the five-sieve to correct these issues we see that the behavior of the green triangle is now correct relative to the boxed prime it represents.



Ex. 19

More importantly we can now see to a very small degree how blocked cells "grow" in their respective vertical columns. For example, as mentioned before, between 21 and 23 on the prime diagonal we see a very open column with only the three sieve but we see that, with the five-sieve applied, there is now a sequence of eight elements partially blocking the green triangle.

As it turns out the growth of the vertical segment from 2 to 8 with the application of the five-sieve is the maximal growth we can expect for the combined three and five sieves.

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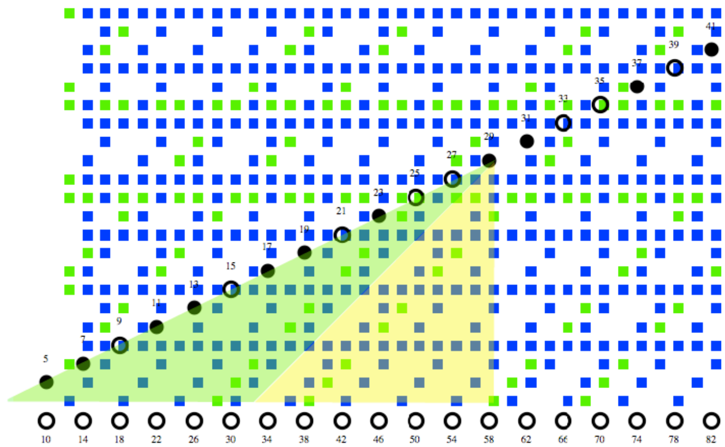
This extending of block columns by adding subsequent sieves is the core of what we are interested in here. Specifically we are interested in precisely characterizing this growth. If we consider that the Goldbach Conjecture operates in a manner similar to the "boxed primes" we can describe the Goldbach Conjecture with a right triangle as show below made up the light green "boxed prime" triangle coupled with the yellow Goldbach triangle to yield a full right triangle represented the full Goldbach Conjecture. The principle difference between the two being the Goldbach triangle works with a larger area than that of the boxed prime for any given prime though effectively both are the same if you consider the base of the green triangle aligned with the lower right corner of the yellow triangle and you look at 1/2 the resulting value (the vertical at 29).


Ex. 20

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Representational Construction

The purpose of this section is to show how to compose a given verticals cells for any even.

First let us consider the tessellation from a different perspective.

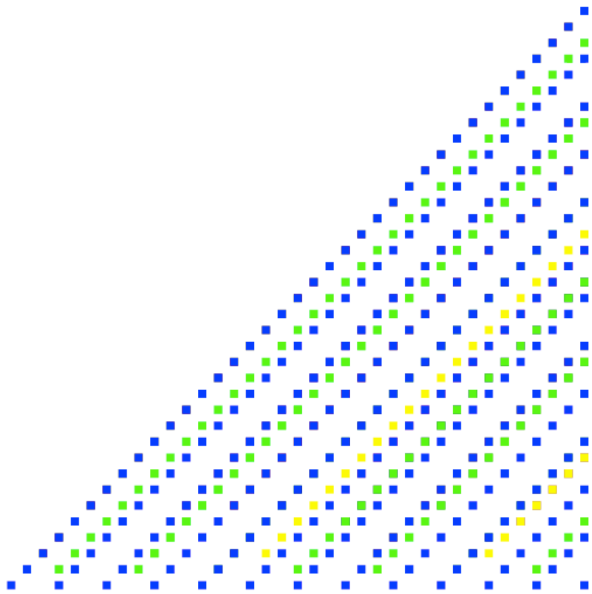
We see that each tessellating tile is made up of essentially two parts: a *diagonal* and a *horizontal*. We imagine splitting the tessellations (such as with Ex. 10 and Ex. 11) into two parts - one that is comprised of only the horizontal elements and one of only the diagonal elements. The cell which is common to both appears in each portion when the splits are shown independently.

So if we consider the *horizontal* portion first we have something that looks like this:


Ex. 21

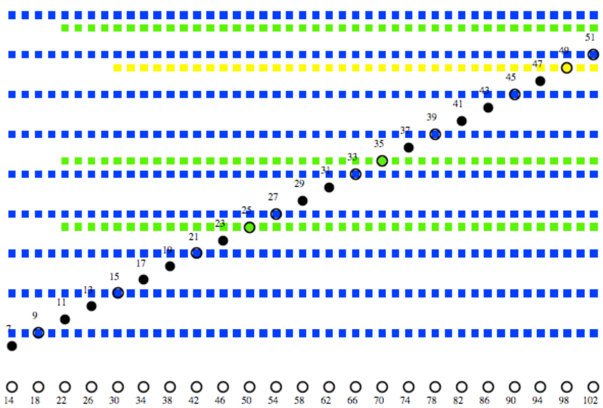
It extends to the *right* and *up*. In terms of *up* it follows the tessellated pattern generated is as in Ex. 12. The more factors added increases the period in which the tessellation repeats. In terms of *right* it simply extends each row infinitely to the right.

Similarly for only the the *diagonal* portions of the tessellations we see:

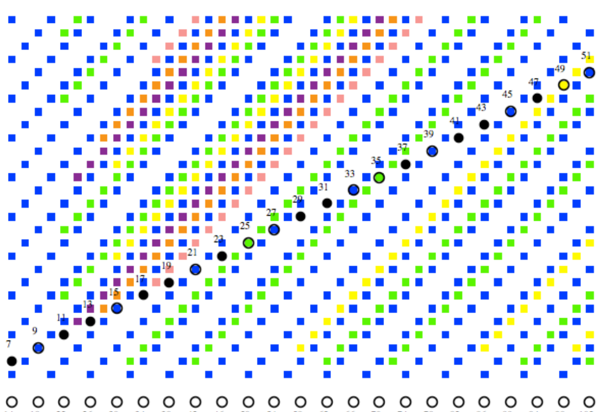


Effectively this is as if the pattern in Ex. 21 rotated 90 degrees clockwise and warped at 45 degrees into a trapezoid. (These examples are not cropped exactly but they should make the concept clear. The examples immediately below have the correct proportions.) We can now observe each component separately as it relates to the prime addends for a given even.

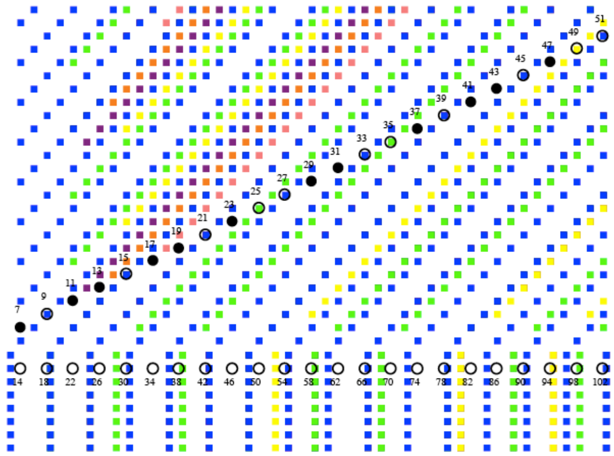
Horizontally we have:



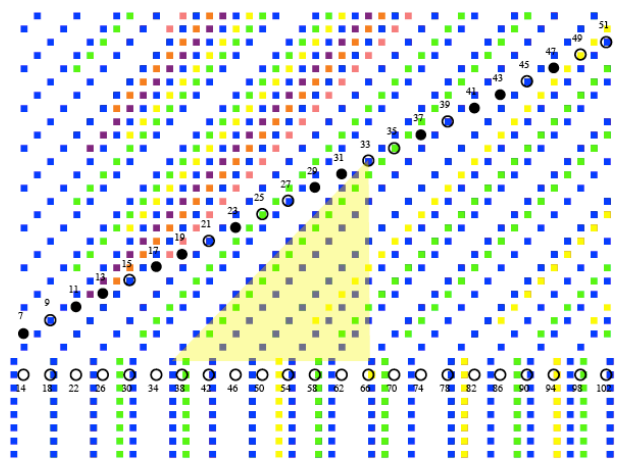
And diagonally we have (the colored squares above the *prime diagonal* are an artifact of the process that creates these images and should be ignored):



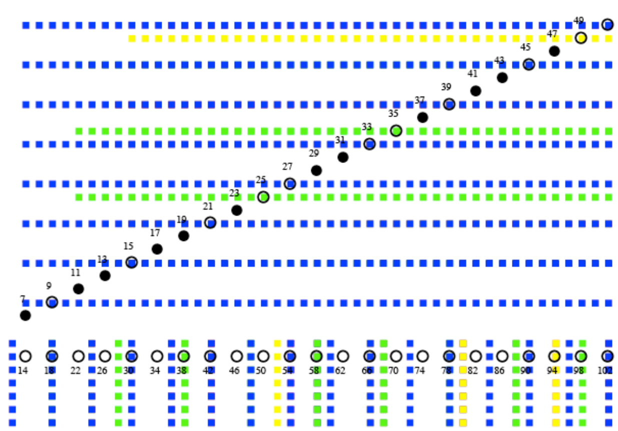
To show the correspondence between the horizontal-only tessellation and the root of the diagonal tessellation we superimpose the horizontal tessellation along the bottom for clarity:



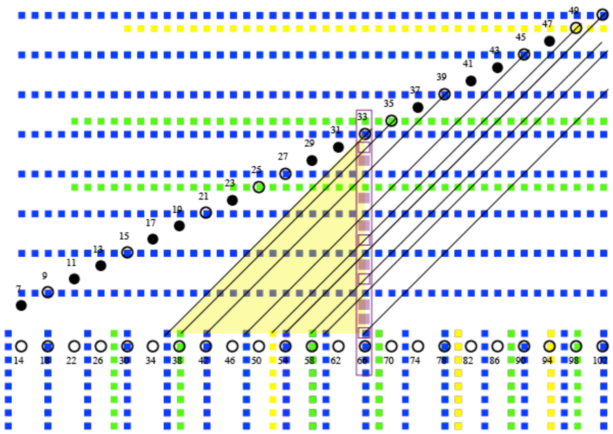
Now, using 66 as an example we will show how the horizontal and diagonal elements interact to construct the prime addends. In the diagram below we have constructed a transparent yellow right triangle with the hypotenuse running from 33 on the prime diagonal to 36 (33 + 3) on the even base. The sides of the triangle run from 36 to 66 and from 66 to 33 on the prime diagonal. (Note some missing small triangles appear in the example below due to the nature of how the image was constructed.) We see that on the side from 33 on the prime diagonal to 66 on the even base various open cells corresponding to *relatively prime addends* of 66 - we say *relatively prime addends* because the horizontal tessellation is not included which will strike additional cells. Hence relative to the prime addends of 66 this is an incomplete picture:



We now show the corresponding structure as above with the diagonal elements replaced by the horizontal ones:



Next we position the yellow transparent triangle in the appropriately corresponding position on the horizontal image. We show where the diagonals from the diagonal image would appear with black lines running from the *even base* to the corresponding position on the *prime diagonal*.



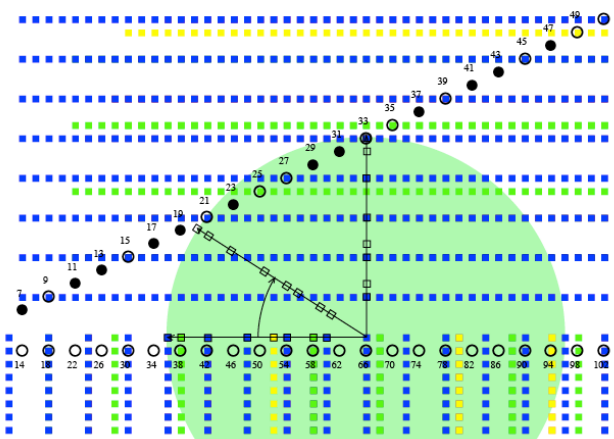
Finally we mark the true *prime addends* of 66 with transparent purple squares. We use open purple squares to show where the addition of the diagonals strike the additional addends.

We now make the following critical observations:

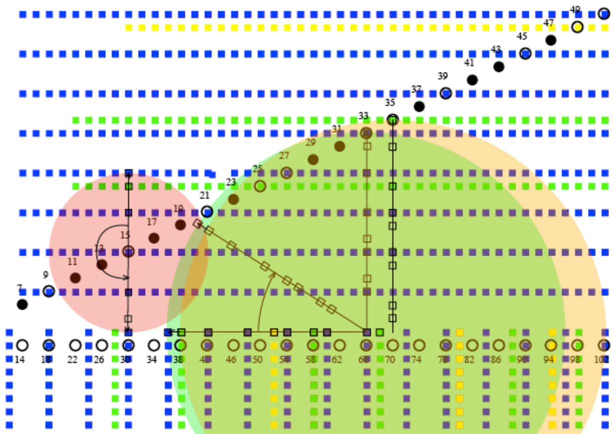
1) The diagonal from 36 on the even base to 33 on the prime diagonal correspond positionally to the same tessellation cycles. This is left to right on the even base starting left of 14 (pattern XXB(lue)XXBXXBXG(reen)BXXB - there are two X's prior to the first blue to account for the position of 3 and 5 on the prime diagonal), and vertically bottom to top on the horizontaly starting at 9 row on the horizontal to 33 on the prime diagonal.

2) We see that the length of the yellow hypotenuse equal to the length of the purple vertical showing in the prime addends (16 in this example).

3) We see that presence of a mark in any cell on the *even base* eliminates the corresponding vertical cell (as traced on the diagonal from one side of the triangle to the other). There are consecutively smaller right triangles where we traverse from the even base to the opposite side by moving up and over one each time.



Finally we can see that "folding" the horizontal tessellation - rotating about the circle at a point on the prime diagonal and carrying the marked cells from above to below - at the prime diagonal yields the prime addends at any intersection of cells where no squares occur (transparent red circle):



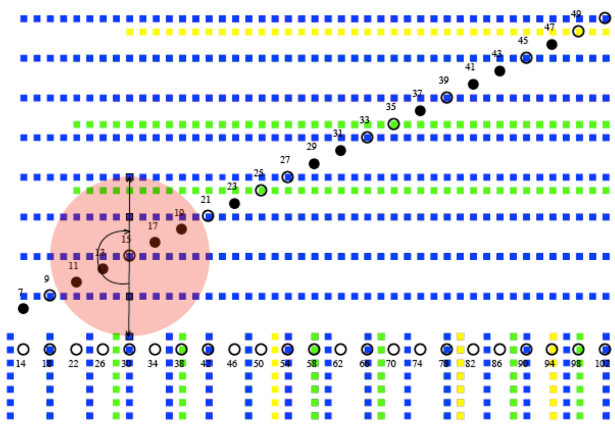
Implications

We can observe that in the case of 15 on the *prime diagonal* the circle is centered on a multiple of three. This means that when the radius is rotated from above the 15 carrying the marked cells to below the 15 only factors greater than three will participate in eliminating addends. In the case of a value such as 25 or 35 which is on centered on a multiple of three the rotation eliminates effectively 1/2 of the open cells due to the relative offset from the multiple of three.

This implies that all factors that align with the center of the circle do not participate in reducing the number of addends after the rotation.

Together this means that for any multiple of three only non-three factors participate in the removal of addends. For non-three multiples the possible positions for prime addends is automatically reduced from 1/3 to 2/3 and the only remaining open cells are affected by factors greater than three.

This seems to imply that if we sweep the circle the opposite way (as see here):



Sweeping the marked cells from below the center to above the only significant additional removals of addends occurs within some epsilon around the center, e.g., consider 15 on the *prime diagonal.*

I believe that this addresses the crux of the matter, at least for factors of 3, is specifically that this reverse sweep can never reduce the number of addends beyond a few around the center of the circle. So the "action", as it where, is within this bound - effectively similar to the "Boxed Primes" discussed earlier.

Goldbach becomes an issue of 1) epsilon - though this I think is bounded by a transition from the inclusion of n - 1 factors to n factors, i.e., 27 - which is bounded and 2) because the primes are infinite there are always gaps for prime addends in the upper portion of the circle.

Sweeps based on non-three factors, e.g., 5, I believe operate as follows: the three-factor cells effectively modulate all results, i.e., leaving only 1/3 of the cells open as potential addends. The remaining factors, being modulated by three, only participate if their centers align with the circle. In this case they can only additionally reduce the number of addends around epsilon.